METRIC SPACES: RE-EXAM 2016

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Problem 1. Let A and B be bounded subsets of a metric space $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ and $A \cap B \neq \varnothing$. Prove the inequality $\mathrm{diam}(A \cup B) \leqslant \mathrm{diam}(A) + \mathrm{diam}(B)$.

(By definition, $\mathrm{diam}(\varnothing) = 0$ and $\mathrm{diam}(S) = \sup_{x,y \in S} \mathrm{d}_{\mathcal{X}}(x, y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Problem 2. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\varnothing \neq A \subseteq \mathcal{X}$ its subset. Prove that the interior $\mathrm{Int}\,(A) = \{a \in A \mid \exists\, \varepsilon(a) > 0,\ \mathrm{B}_{\varepsilon}^{\mathrm{d}_{\mathcal{X}}}(a) \subseteq A\}$ is open in $\mathcal{X}$.

Problem 3. Let $\mathcal{X}$ be a space such that every continuous function $f\colon \mathcal{X} \to \mathbb{R}$ has the following property: if $a < c < b$, $f(x) = a$, and $f(y) = b$, then there exists $z \in \mathcal{X}$ such that $f(z) = c$. Prove $\mathcal{X}$ is connected.

(The set $\mathbb{R}$ is equipped with the standard Euclidean topology.)

Problem 4. Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a sequentially compact space $\mathcal{X}$ and $V_n \supseteq V_{n+1}$. Prove that

$$\bigcap_{n=1}^{+\infty} V_n \neq \varnothing.$$

- Is this intersection always non-empty if the hypothesis of sequential compactness is discarded? (state and prove, e.g., by counterexample)

Problem 5. Prove that the algebraic equation $20\,x = 1 - x^{16}$ has a unique solution in the segment $[0, 1] \subset \mathbb{R}$.

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Each problem value = 20% = 10% + 10% of the exam. GOOD LUCK!